MEscope Application Note 28

Mathematics of a Mass-Spring-Damper System

The steps in this Application Note can be carried out using any MEscope package that includes the **VES-3600 Advanced Signal Processing** and **VES-4000 Modal Analysis** options. Without these options, you can still carry out the steps in this App Note using the **AppNote28** project file. These steps might also require a ***more recent release date*** of MEscope.

APP NOTE 28 PROJECT FILE

- To retrieve the Project file for this App Note, **click here** to download **AppNote28.zip**

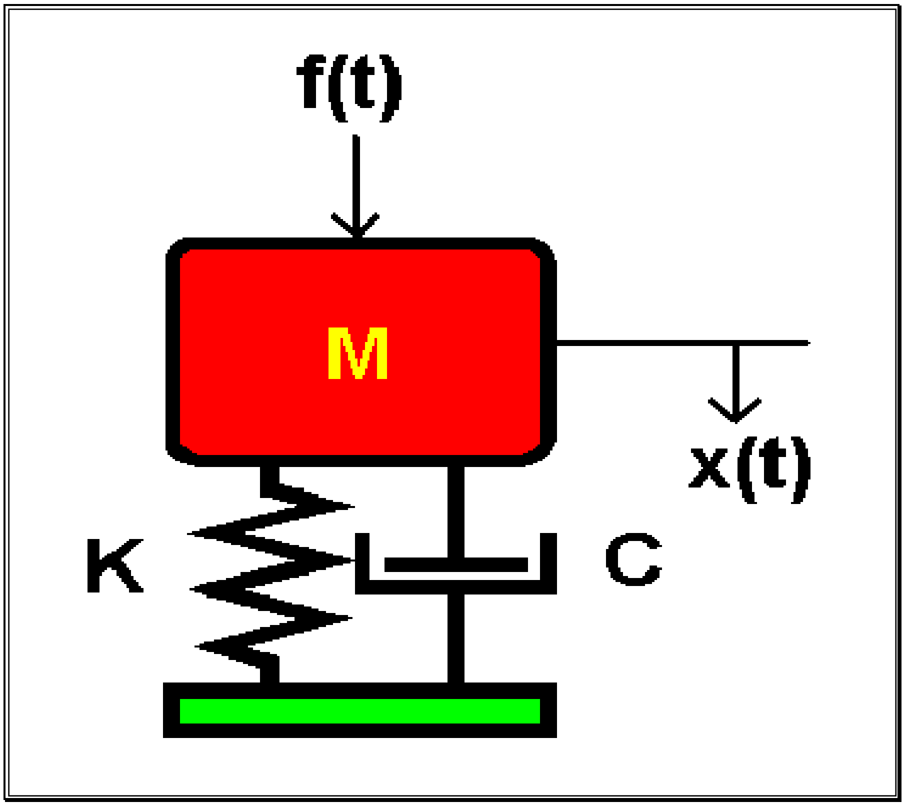
This Project contains ***numbered Hotkeys & Scripts*** of commands for carrying out the steps of this App Note.

- ***Hold down*** **the Ctrl key** and ***click on a*** **Hotkey** to open its Script window

INTRODUCTION

In this note, MEscope is used to explore the properties of the mass-spring-damper system shown in the figure below. Its equation of motion will be solved for its mode of vibration. Then, an **FRF** will be synthesized using its mode shape, and its stiffness and mass lines will be examined.

Then, the **FRF** will be curve fit to extract its modal parameters. Finally, we will look at how the modal parameters are contained in the Impulse Response Function, the Inverse FFT of the **FRF**.

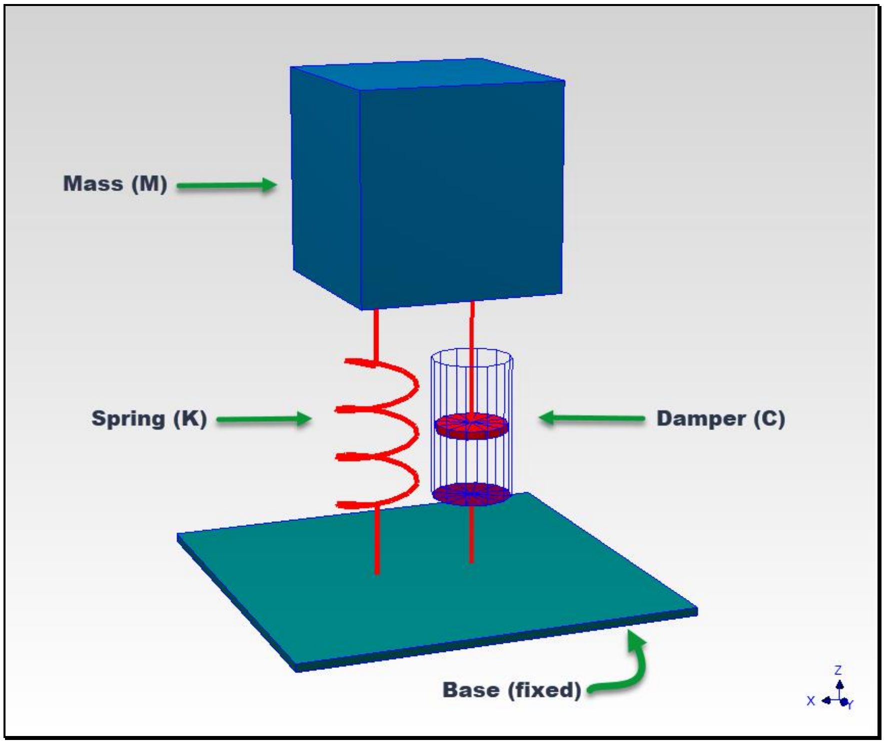


Mass-Spring-Damper.

The purpose of this Application Note is to review the details of modal analysis, to provide a better understanding of the modal properties of all structures. The modal properties of real-world structures are analyzed using a multi-degree-of-freedom (MDOF) dynamic model, whereas the model used here is a single degree-of-freedom (**SDOF**) model. Nevertheless, the dynamics of MDOF structures are better understood by analyzing the dynamics of this **SDOF** structure.

Modes are defined for structures, the dynamics of which can be represented by ***linear ordinary differential equations*** like the one in the **Background Math** section below. The dynamic behavior of the mass-spring-damper structure in the figure above is represented by a single (scalar) equation. An MDOF structure is represented by multiple equations, which are written in matrix form.

Because of the ***superposition property*** of linear systems, the dynamics of an MDOF structure can be written as a ***summation of contributions due to each of its mode shapes***. Each mode shape can be thought of as representing the dynamics of a single mass-spring-damper system.



Mass-Spring-Damper 3D Model.

BACKGROUND MATH

The time domain equation of motion for the mass-spring-damper is represented by ***Newton's Second Law***, written as the following ***force balance*** on a structure,

$$\mathbf{M}\,\ddot{\mathbf{x}}(\mathbf{t}) + \mathbf{C}\,\dot{\mathbf{x}}(\mathbf{t}) + \mathbf{K}\,\mathbf{x}(\mathbf{t}) = \mathbf{f}(\mathbf{t})$$

$\mathbf{M}$ ➔ Mass

$\mathbf{C}$ ➔ Damping

$\mathbf{K}$ ➔ Stiffness

$\ddot{\mathbf{x}}(\mathbf{t})$ ➔ acceleration

$\dot{\mathbf{x}}(\mathbf{t})$ ➔ velecity

$\mathbf{x}(\mathbf{t})$ ➔ displacement

$\mathbf{f}(\mathbf{t})$ ➔ excitation force

LAPLACE TRANSFORMS

By taking Laplace transforms of the terms in the differential equation above and setting initial conditions to zero, an equivalent frequency domain equation of motion results,

$$[\mathbf{M}\ \mathbf{s}^2 + \mathbf{C}\ \mathbf{s} + \mathbf{K}]\ \mathbf{X}(\mathbf{s}) = \mathbf{F}(\mathbf{s})$$

X(s) ➔ Laplace transform of the displacement

F(s) ➔ Laplace transform of the force

$\mathbf{s} = \boldsymbol{\sigma} + \mathbf{j}\boldsymbol{\omega}$ ➔ complex Laplace variable

TRANSFER FUNCTION

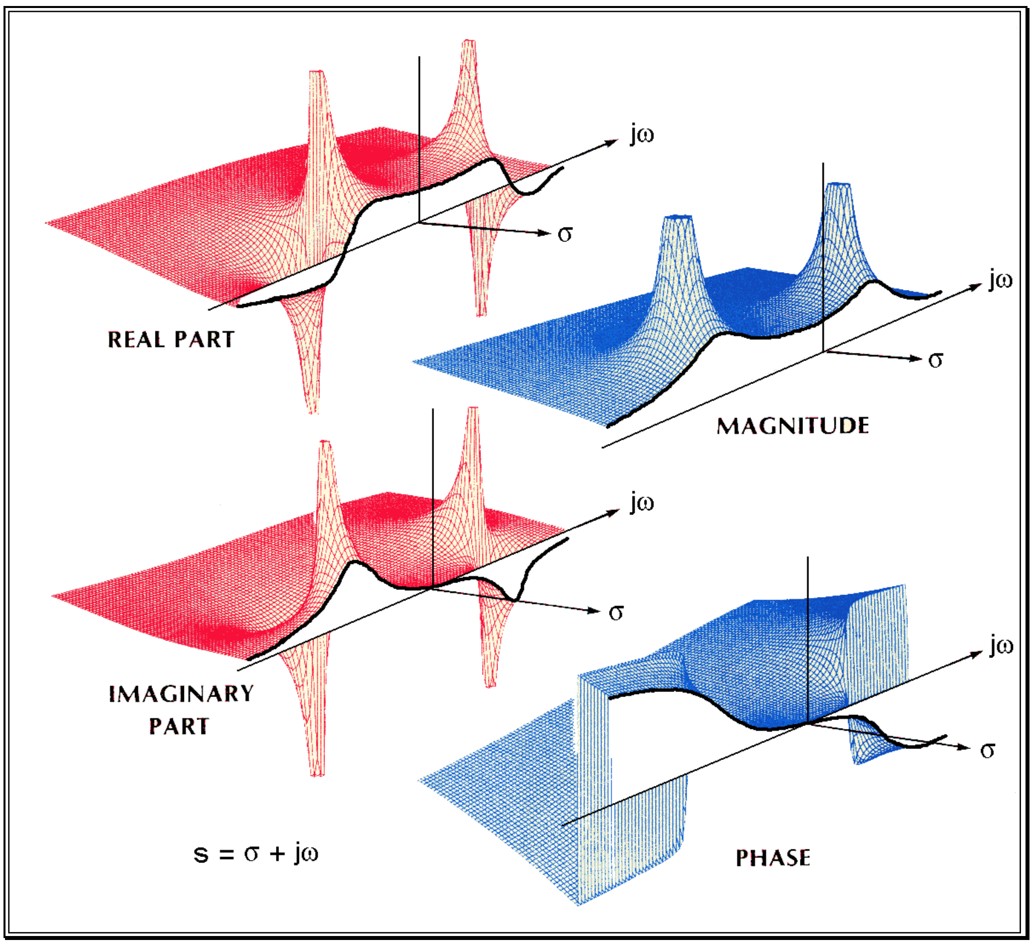
The above equation can be rewritten by simply dividing both sides by the coefficients of the left-hand side.

$$\mathbf{X}(\mathbf{s}) = \left(\frac{\mathbf{1}}{\mathbf{M}\mathbf{s}^2 + \mathbf{C}\ \mathbf{s} + \mathbf{K}}\right)\ \mathbf{F}(\mathbf{s})$$

The new coefficient on the right-hand side of the above equation is called the **Transfer Function**,

$$\mathbf{H}(\mathbf{s}) = \frac{\mathbf{X}(\mathbf{s})}{\mathbf{F}(\mathbf{s})} = \left(\frac{\mathbf{1}}{\mathbf{M}\mathbf{s}^2 + \mathbf{C}\mathbf{s} + \mathbf{K}}\right)$$

The Transfer Function is ***complex valued*** and has ***real & imaginary*** parts or equivalently ***magnitude & phase***. The two parts of the Transfer Function can be plotted on the complex Laplace plane (or **s**-plane), as shown below.



Transfer Function Plotted Over Half of the S-Plane.

POLES OF THE TRANSFER FUNCTION

Notice that the magnitude of the Transfer Function ***has two peaks***. These are points where the value of the Transfer Function ***goes to infinity***. The ***real & imaginary parts*** also have the ***same two peaks***.

The Transfer Function goes to infinity for values on the **s**-Plane where its ***denominator is zero***. It is also clear that ***as s goes to infinity***, the Transfer Function ***approaches zer***o.

CHARACTERISTIC POLYNOMIAL

The denominator of the Transfer Function is a ***second order polynomial*** in the **s** variable and is called the ***characteristic polynomial***. Since it is a second order polynomial, it has two roots (values of **s** for which ***it is zero)***.

These two roots of the denominator are called the ***poles*** of the Transfer Function.

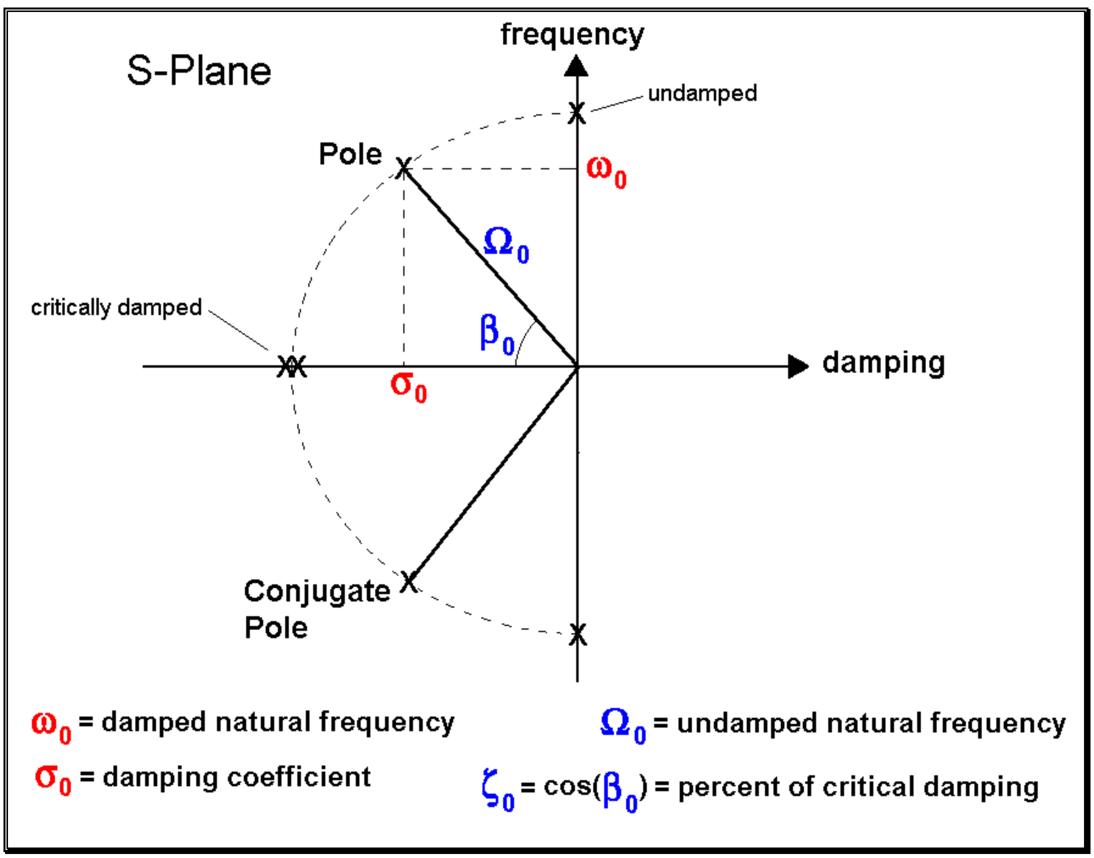
The poles are ***complex conjugates*** of one another. The poles are the locations on the **s**-plane where the Transfer Function has a ***value of infinity***. The poles are also called ***eigenvalues***.

$$\mathbf{p_0 = -\sigma_0 + j\omega_0 \ , p_0^* = -\sigma_0 - j\omega_0}$$

S-PLANE NOMENCLATURE

The ***real axis*** in the **s**-Plane is called the ***damping axis***, and the ***imaginary axis*** is called the ***frequency axis***.

The locations of the poles in the **s**-Plane have also been given some other commonly used names, as shown below.



S-Plane Nomenclature.

MODAL PARAMETERS

The coordinates of the poles in the **s**-Plane are also modal parameters. The Transfer Function can be written in terms of its pole locations, or modal parameters,

$$\mathbf{H(s)} = \left(\frac{\mathbf{1/M}}{\mathbf{s^2 + 2\sigma_0 s + \Omega_0{}^2}}\right)$$

$$\boldsymbol{\sigma_0} = \frac{\mathbf{C}}{\mathbf{2\ M}}\ , \boldsymbol{\Omega_0}^{\mathbf{2}} = \frac{\mathbf{K}}{\mathbf{M}}$$

$$\boldsymbol{\Omega_0}^{\mathbf{2}} = \boldsymbol{\sigma_0}^{\mathbf{2}} + \boldsymbol{\omega_0}^{\mathbf{2}}$$

$\boldsymbol{\sigma_0}$ ➔ modal damping

$\boldsymbol{\Omega_0}$ ➔ un-damped modal frequency

$\boldsymbol{\omega_0}$ ➔ damped modal frequency

The **percent of critical damping** ($\boldsymbol{\zeta_0}$) is written as,

$$\boldsymbol{\zeta_0} = \frac{\boldsymbol{\sigma_0}}{\boldsymbol{\Omega_0}} = \frac{\mathbf{C}}{\mathbf{2\sqrt{MK}}}$$

FREQUENCY RESPONSE FUNCTION (FRF)

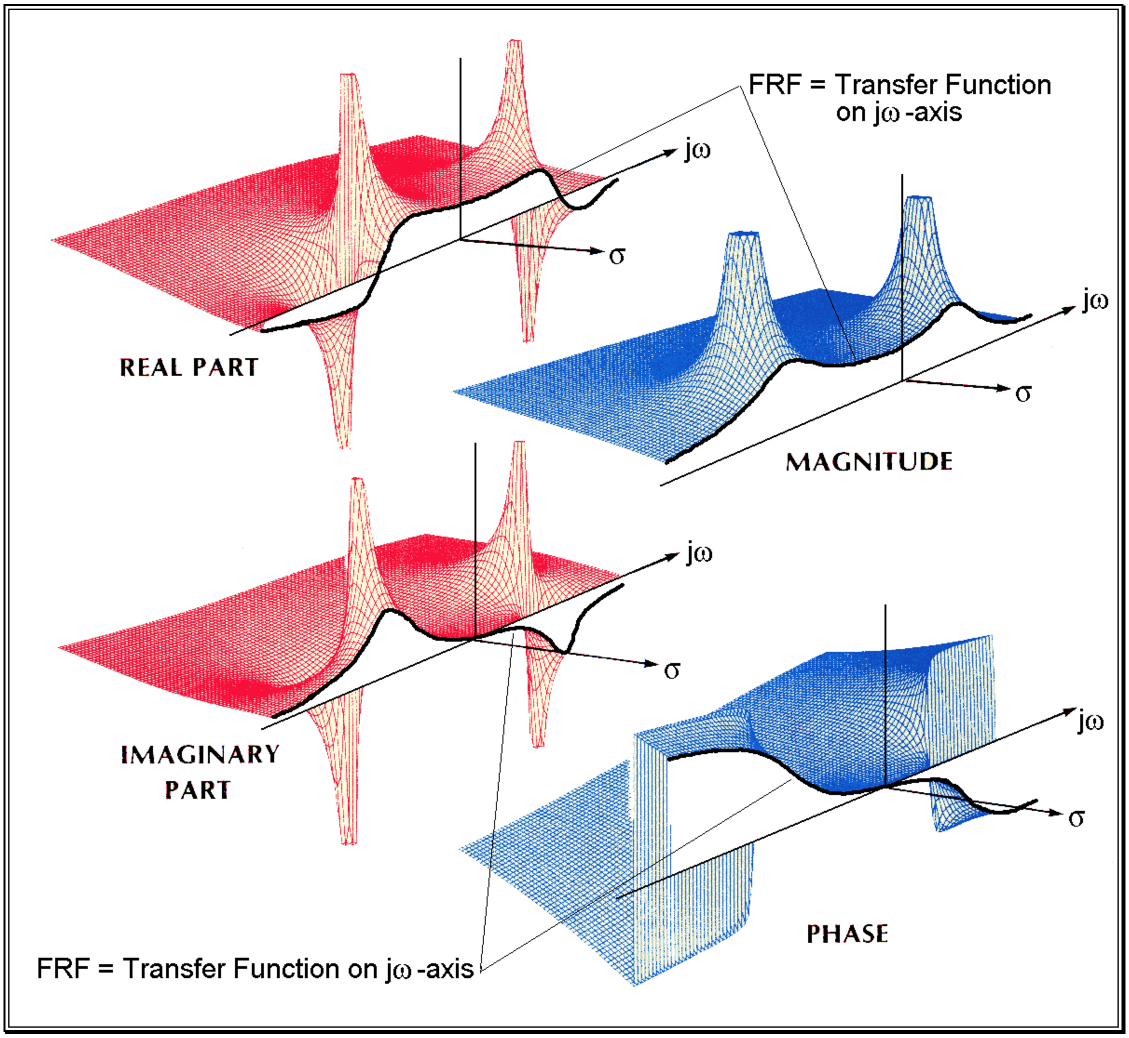
In the figure below, ***the Transfer Function has only been plotted for half of the s-Plane.*** It has only been plotted for ***negative values of σ*** (the real part of **s**). This was done so that the values of the Transfer Function along the **jω**-axis (the imaginary part of **s**) are clearly seen.

The Frequency Response Function (**FRF**) is the values of the Transfer Function *along the* jω-axis.

The **FRF** values along the **jω-**axis are shown with black lines below. Since the **FRF** is only defined along the **jω-**axis, **s** can be replaced by its imaginary part **jω,**

$$\mathbf{FRF = H(j\omega) = H(s)\Big|_{s=j\omega}}$$

$$\mathbf{= \frac{X(s)}{F(s)}\Bigg|_{s=j\omega} = \frac{X(j\omega)}{F(j\omega)}}$$



***FRF** Plotted on the* **jω-axis**.

FRF IN PARTIAL FRACTION FORM

The **FRF** for an **SDOF** (mass-spring-damper) can now be written in terms of modal parameters by also replacing the **s**-variable with **jω** in the equation for the Transfer Function in terms of modal parameters,

$$\mathbf{H(j\omega)} = \frac{\mathbf{(1/M)}}{\mathbf{(j\omega)^2 + 2\sigma_0\ j\omega + \Omega_0{}^2}}$$

Furthermore, using the poles of the Transfer Function, the FRF can be written in a partial fraction expansion form,

$$\mathbf{H(j\omega)} = \frac{\mathbf{1}}{\mathbf{2j}}\left[\frac{\mathbf{R_0}}{\mathbf{j\omega - p_0}} - \frac{\mathbf{R_0}}{\mathbf{j\omega - p_0^*}}\right]$$

$$\mathbf{R_0 = 1/\omega_0 M}$$

$\mathbf{R_0}$ ➔ **modal residue**

$$\mathbf{p_0 = -\sigma_0 + j\omega_0, p_0^* = -\sigma_0 - j\omega_0}$$

The **modal residue** is the ***amplitude (or strength)*** of the numerator of each resonance term in the above equation.

The **FRF** of an **SDOF** is the ***summation of two resonance curves***, each one with a peak near a pole location.

The Partial Fraction Form of the FRF for an **SDOF** is fully represented by two poles and two residues.

Since the residues are equal and the poles are complex conjugates of one another, the complete dynamics of an **SDOF** system is ***fully represented by a modal frequency*** $(\mathbf{j\omega_0})$, ***modal damping*** $(\boldsymbol{\sigma_0})$, and ***modal residue*** $(\mathbf{R_0})$.

MODE SHAPES

One final step is to represent the **FRF** in terms of mode shapes instead of residues,

$$\mathbf{H(j\omega)} = \frac{\mathbf{1}}{\mathbf{2j}}\left[\frac{\{\mathbf{u_0}\}^2}{\mathbf{j\omega - p_0}} - \frac{\{\mathbf{u_0}\}^2}{\mathbf{j\omega - p_0^*}}\right]$$

$$\{\mathbf{u_0}\} = \begin{Bmatrix} \frac{\mathbf{1}}{\sqrt{\mathbf{A\omega_0 M}}} \\ \mathbf{0} \end{Bmatrix},\ \mathbf{A} \Rightarrow \text{scaling constant}$$

The mode shape $\{\mathbf{u_0}\}$is a vector. Its first component is the ***square root of the residue***, and its second component is zero. The second component ***corresponds to the ground***, where there is no motion.

The mode shape also contains a scaling constant (**A**).

A mode shape ***does not have unique values***. Only its ***shape is unique***, one component relative to another.

Because it does not have unique values, a mode shape is also ***called an eigenvector***.

The dynamics of an **SDOF** is fully ***represented by two poles (eigenvalues)*** and ***two mode shapes (eigenvectors)***.

Because of symmetry, the dynamics of an **SDOF** *is **fully represented*** by a **modal frequency** $(\mathbf{j\omega_0})$, **modal damping** $(\boldsymbol{\sigma_0})$, and a **mode shape** $\{\boldsymbol{u_0}\}$

IMPULSE RESPONSE FUNCTION (IRF)

An Impulse Response Function is the **Inverse FFT** of an **FRF**.

When it is written in terms of modal parameters, an **IRF** provides ***the best source of meaning*** for modal parameters.

In terms of modal parameters, the **IRF** is written,

$$\mathbf{h(t)} = \mathbf{FFT^{-1}}\left(\frac{\mathbf{1}}{\mathbf{2j}}\left[\frac{\mathbf{R_0}}{\mathbf{j\omega - p_0}} - \frac{\mathbf{R_0}}{\mathbf{j\omega - p_0^*}}\right]\right)$$

$$\mathbf{h(t)} = \frac{\mathbf{T}}{\mathbf{2j}}\left[\mathbf{R_0 e^{p_0 t}} - \mathbf{R_0 e^{p_0^* t}}\right]$$

$$\mathbf{h(t)} = \mathbf{T}\ |\mathbf{R_0}|\mathbf{e^{-\sigma_0 t}}(\mathbf{sin}(\,\boldsymbol{\omega_0}\mathbf{t} + \boldsymbol{\alpha_0}))$$

$$\boldsymbol{\alpha_0} \Rightarrow \text{angle of } \mathbf{R_0}$$

The equation above shows the role that each modal parameter plays in the **IRF**.

$\boldsymbol{\omega_0}$ ➔ multiplies the time variable (**t**) in the sinusoidal function $(\sin(\omega_0 t + \alpha_0))$

$\boldsymbol{\omega_0}$ ➔defines the frequency of oscillation, hence $\boldsymbol{\omega_0}$is called ***damped modal frequency***

$\boldsymbol{\sigma_0}$ ➔ the coefficient in the exponential term $(e^{-\sigma_0 t})$ that defines the ***envelope of decay*** of the **IRF**

$\boldsymbol{\sigma_0}$ ➔ called the ***damping decay constant***, ***modal damping coefficient***, ***half power point damping, 3 dB point damping***

In a real-world structure, vibration decay is caused by a ***combination of damping mechanisms*** within or about the structure.

On earth, the ***most common damping mechanism*** is the ***surrounding air***.

FEA MODEL

In MEscope, a graphical 3D model is needed to display experimentally derived ODS's and mode shapes in animation. The mass-spring damper model above was created in **App Note 07A** using ***five substructures***. Building a model using substructures makes setting up the model for shape animation and FEA modeling much easier.

In this App Note, an FEA mass, spring, & damper ***have already been added*** to the 3D model.

The resulting FEA model will be solved for its single mode of vibration ***with and without the damper*** included.

The FEA models of most real-world structures ***do not include damping*** because ***it is too difficult*** to include an accurate damping model of the structure into a set of MDOF (multiple degree-of-freedom) differential equations. When damping is not included, the FEA model can still be solved for its mode shapes.

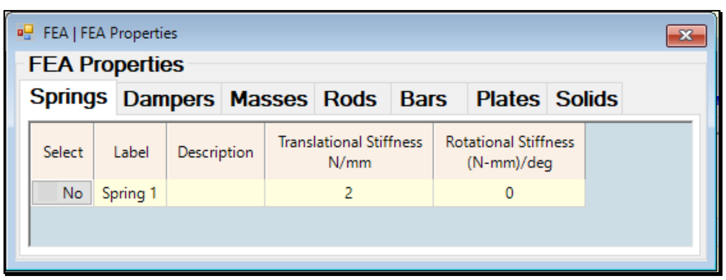
Mode shapes of an **undamped structure** are called **normal modes**.

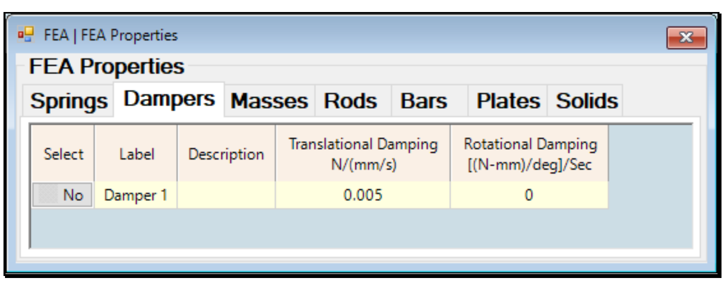
FEA MASS, STIFFNESS & DAMPING

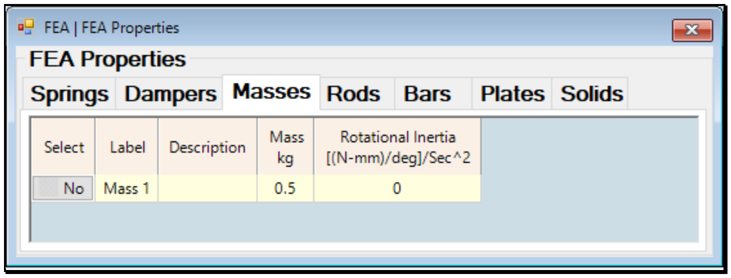
The FEA properties of the Mass, Spring & Damper have already been added to the FEA Properties dialog. To display the FEA Properties,

- Open an **STR** window and execute **FEA | FEA Properties**

The following properties are displayed on the **Springs**, **Dampers**, and **Masses** tabs.







MODAL FREQUENCY & DAMPING

Millimeter (**mm**) units are internally converted to Meters (**m**) to make them consistent with Newtons (**N**) and kilograms (**kg**).

The **un-damped natural frequency** ($\mathbf{f_n}$) of the Mass-Spring-Damper is calculated with the following formula

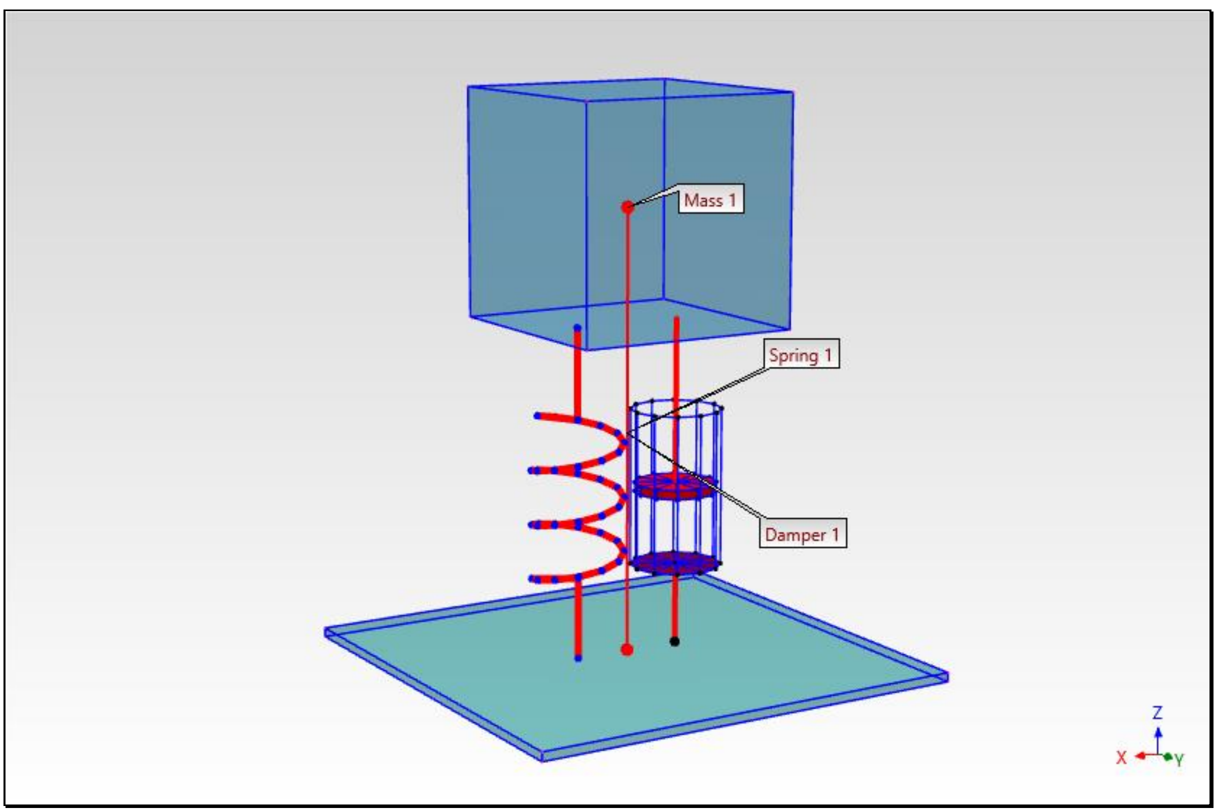
$$\mathbf{f_n = \frac{1}{2\pi}\sqrt{\frac{K}{M}} = \frac{1}{2\pi}\sqrt{\frac{2000}{0.5}} = 10.07\ Hz}$$

The **percent of critical damping** (ζ) is calculated with the following formula,

$$\boldsymbol{\zeta} = \mathbf{\frac{C}{\sqrt{4KM}} = \frac{0.005}{\sqrt{4 \times 2000 \times 0.5}} = 7.906\ \%}$$

The **damped natural frequency** ($\mathbf{f_d}$) is calculated with the following formula,

$$\mathbf{f_d = f_n\sqrt{1-\zeta^2} = 10.03\ Hz}$$



STEP 1 – DAMPED AND UNDAMPED MODE SHAPE

- ***Press*** **Hotkey 1 Damp vs. Undamped Mode Shape**

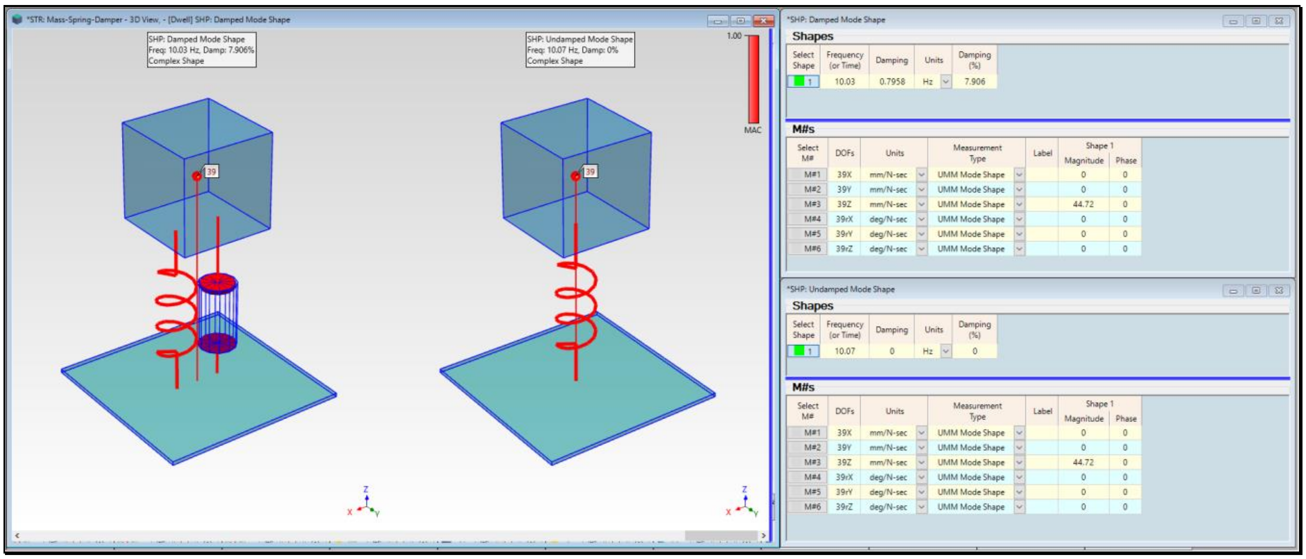
In this step, the FEA mode shape of the **damped SDOF model** is compared with the mode shape of the **undamped SDOF model**.

The Mass, Spring & Damper have already been attached to the 3D model

The Mass is attached to a Point in the center of the mass cube

The Spring & Damper have been attached between the Mass Point and a Fixed base Point. All the Points on the Base plate are Fixed

After the undamped and damped FEA mode shapes have been calculated, they are displayed side-by-side in animation, as shown below.



Damped & Undamped Mode Shapes Side-by-Side.

MAC ➔ 1.0 between the two mode shapes, meaning that ***they are identical***. That is also apparent by observing the **M#s** in each Shape Table ***on the right***. The FEA solver also calculated rotational DOFs but those are not displayed in animation of the **SDOF** models.

The modal frequencies, **damped ➔ 10.03 Hz**, **undamped ➔ 10.07 Hz**, are identical to the values calculated with the analytical formulas.

UMM MODE SHAPES

In the two Shape Tables listed above, the M#s of the FEA mode shapes are designated as **UMM** Mode Shapes.

In MEscope mode shapes scaled to Unit Modal Masses are called **UMM** mode shapes.

A common way to scale mode shapes is so that they yield unit modal masses. If the mass is pre- and post-multiplied by a **UMM** mode shape, the **resulting modal mass ➔ 1**. For an MDOF system, if the mass matrix is pre- and post-multiplying by a set of **UMM** mode shapes the resulting modal mass matrix **➔1 *along the diagonal.***

When FEA mode shapes are calculated in MEscope, the resulting mode shapes are **UMM** mode shapes.

THE DRIVING POINT FRF

An **FRF**, like a Transfer Function, defines the dynamic characteristics between two DOFs of a structure. MDOF systems have many DOF pairs for which FRFs can be calculated. The **SDOF** system in this App Note has only one meaningful driving point at **DOF 39Z**. Force can be applied at the mass point (**Point 39**) in the vertical (**Z**) direction. Using the damped **UMM** mode shape an **FRF** can be synthesized between the DOF (**39Z**) and itself.

Any **FRF *between a* DOF *and itself*** is called a driving point **FRF**.

Since an **FRF** is a ratio of response (**Output**) divided by excitation (**Input**). In this case, the units of the driving point **FRF** are (**mm/N**), and its DOFs are denoted **39Z:39Z**.

DRIVING POINT RESIDUE

UMM mode shapes are ***scaled differently*** than **Residue** mode shapes. For an **SDOF** model, the driving point Residue is equated to ***the driving point component*** of the **UMM mode shape** by the formula,

$$\mathbf{R_0} = \frac{\{\mathbf{U_0}\}^2}{\mathbf{A\varpi_0 M}}$$

A **Residue** mode shape ***has unique values*** and engineering units, in this case (**mm/(N-sec)**)

A **UMM** mode shape ***has no unique values***. ***Its shape is unique, but its values are not***

Since the scaling constant (**A**) of a mode shape is arbitrary, ***it can always be equal to* 1**

With **A=1**, the driving point component of the **UMM** mode shapes in the previous figure can be scaled to driving point Residues

$$\mathbf{R_0} = \frac{\{\mathbf{U_0}\}^2}{\mathbf{A\varpi_0 M}} = \frac{\{\mathbf{44.72}\}^2}{\left((\mathbf{2\ \pi})\mathbf{10}\right)(\mathbf{0.5})} \approxeq \mathbf{31.72}$$

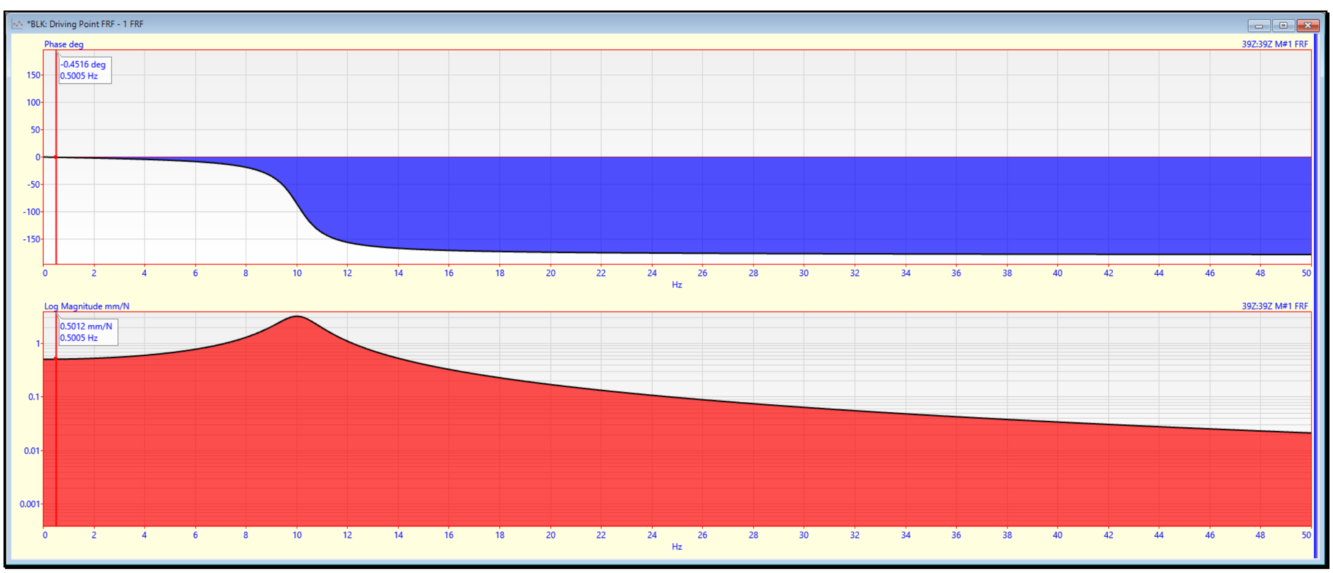
To verify this result,

- Execute **Tools | Scaling | UMM to Residue Shapes** and ***select* DOF 39Z**, the driving point DOF at the Mass

STEP 2 - SYNTHESIZING THE DRIVING POINT FRF

- ***Press* Hotkey 2 Driving Point FRF**

The driving Point **FRF** will be displayed as shown below.



*Driving Point **FRF 39Z:39Z** in (**mm/N**) Units.*

STIFFNESS LINE OF THE FRF

As the frequency approaches **0 Hz**, the equation for the **FRF** shows that a (displacement/force) **FRF** approaches a constant, equal to the ***inverse of the stiffness***.

The ***inverse of the*** **stiffness** is called the **flexibility**.

By letting **S=0** in the Transfer Function equation, the flexibility is,

$$H(0) \approx \frac{1}{K} = \frac{1}{2} = 0.5\ \text{(mm/N)}$$

- The flexibility is verified by the cursor value on the Log Magnitude in the figure above.

STEP 3 - MASS LINE OF THE FRF

- ***Press* Hotkey 3 Mass Line of the FRF**

As frequency becomes large, the **FRF** becomes dominated by the $j\omega^2$ term in its denominator, or characteristic polynomial. Therefore, at high frequencies the magnitude of the **FRF** can be approximated by,

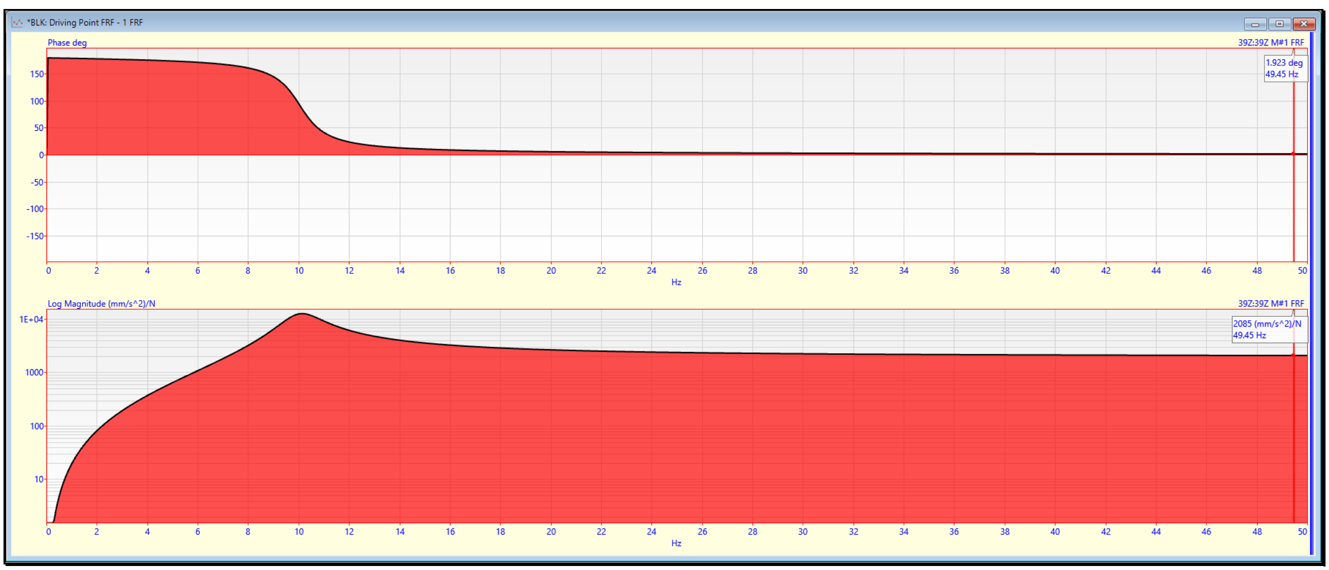
$$|H(\omega)| \approx \left(\frac{1}{M\omega^2}\right)$$

When **Hotkey 3 is *pressed***, the **Tools | Differentiate** command ***is executed twice*** to multiply the **(displacement/force) FRF by** ω^2. Hence, its magnitude for high frequencies becomes a constant,

$$\omega^2\ |H(\omega)| \approx \left(\frac{1}{M}\right)$$

The cursor value on the Log Magnitude below is ***approximately*** **2 (m/sec)^2/N ➔ 0.5 Kg**, the mass of the Mass-Spring-Damper structure.

One Newton ➔ (one kilogram) x (one meter per second squared).



Line Cursor Value Showing the Value of the Mass Line.

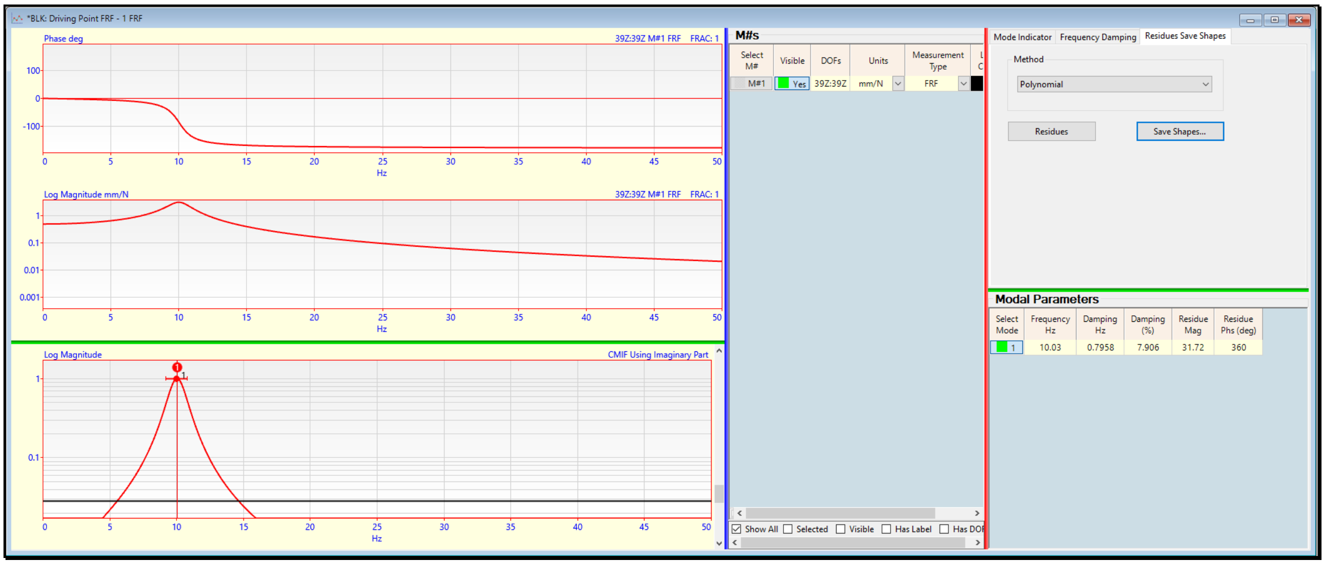
STEP 4 - QUICK FIT OF THE FRF

- ***Press* Hotkey 4 Quick fit of the FRF**

A Driving Point **FRF** of the **SDOF** is a complete representation of the dynamics of the mass-spring-damper system at its driving point.

Curve fitting an **FRF** will recover the modal parameters of the **SDOF**, which are also ***a complete representation of the driving point dynamics***.

The modal parameter estimates are displayed in the **Modal Parameters** spreadsheet shown below ***on the lower right***.



*Curve Fitting the Synthesized **FRF**.*

The modal **frequency**, **damping** & **residue** values are all the correct values of the damped **SDOF** system.

IMPULSE RESPONSE FUNCTION (IRF)

The **IRF** is the product of two functions, an ***exponential decay function*** ($\mathbf{T}\ |\mathbf{R_0}|\mathbf{e^{-\sigma_0 t}}$) and a ***sinusoidal function*** $(\mathbf{sin(\omega_0 t + \alpha)})$. The exponential decay $(\mathbf{e^{-\sigma_0 t}})$ and sinusoidal function ***are dimensionless***.

Only **T** and $\mathbf{R_0}$ have units

IRF units ➔ (**Seconds**) x (**Residue units**)

IRF units ➔ **FRF units**

ENVELOPE OF DECAY

The ***maximum possible magnitude*** of the **IRF** is the value of the exponential function for **t=0**.

$$\mathbf{max.\ \ mag. = T\ |R_0|e^{-\sigma_0 t}|_{t=0} = T\ |R_0|}$$

This is the product of the time length (**T**) of the **IRF**, and the magnitude of the residue $|\mathbf{R_0}|$.

To check the envelope of the **IRF**, a new **FRF** will be synthesized using the modal parameters, but this time the **frequency increment** will be chosen so that **T=1**. One of the equations governing the FFT is,

$$\mathbf{\Delta f = \frac{1}{T} = 1\ Hz}$$

Therefore, it an **FRF** is synthesized with **Δf ➔ 1**, and the ***Inverse FFT*** is applied to it, an **IRF** with **T=1** will result.

STEP 5 - LOG DECREMENT OF THE IRF

- ***Press* Hotkey 5 Log Decrement of the IRF**

The **IRF** shown below is displayed with the Line cursor at the maximum peak.

Peak value of the **IRF ➔ 28.07 mm/N**

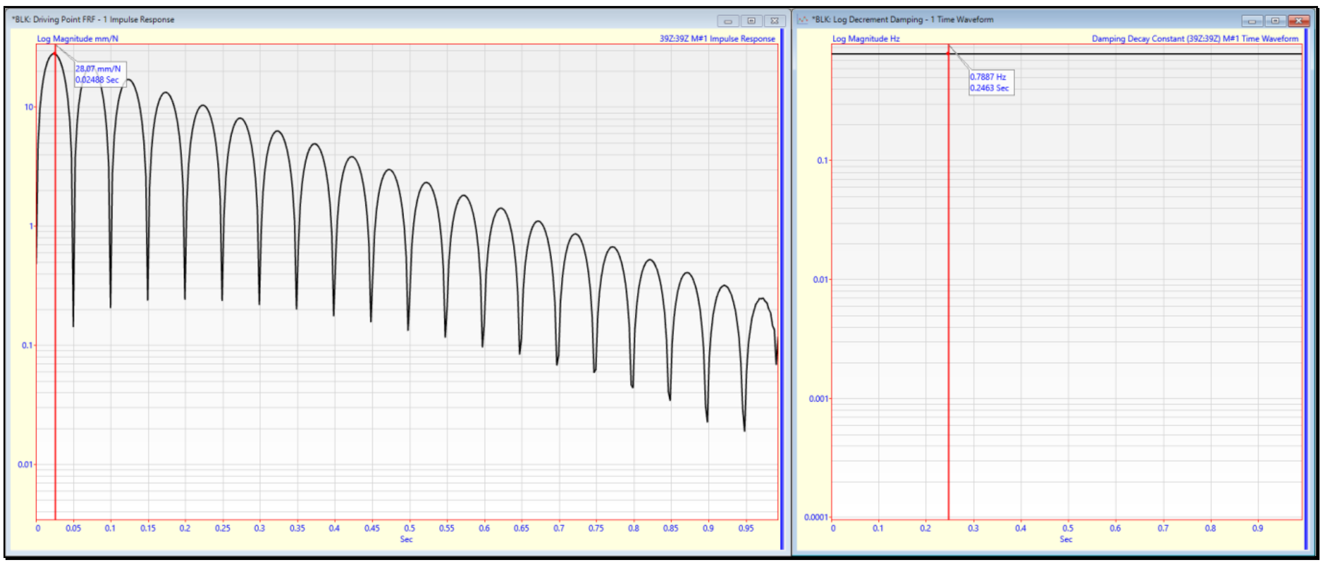
Since **T=1**, the maximum envelope value (for **t = 0**) should be ***approximately equal*** to the ***magnitude of the residue***,

R_0 ➔ 31.72 mm/N-sec

Another Data Block with the value of the modal damping (in **Hz**) is displayed ***on the right***.

This is the **Log Decrement**, calculated as the slope of the peaks in the the Log Magnitude of the **IRF**.

Log Decrement **➔ 0.7887 ➔ *closely matches* 0.7958 Hz**



Log Decrement of the IRF.

STEP 6 - REVIEW STEPS

To review the steps of this App Note,

- ***Press* Hotkey 6 Review Steps**