MEscope Application Note 10

Averaging DFTs, Auto Spectra & Cross Spectra

INTRODUCTION

In MEscope, both the **Digital Fourier Transform (DFT)** and **Cross or Auto** spectra can be calculated from time domain signals. These calculations can include spectrum averaging, which includes choices of time domain window (**Hanning, Flat Top**, or **Rectangular**), **Number of Averages**, **Peak Hold or Stable** averaging, and **Overlap Processing**.

Spectrum averaging is normally done to **reduce or remove random noise and non-linear effects** from spectral estimates.

It will be shown in this Application Note that noise is removed differently from a **DFT**, a **Cross** spectrum, and an **Auto** spectrum.

The **DFT** is calculated by applying the **FFT** algorithm to a **uniformly-sampled digital time waveform**.

A **Auto** spectrum is calculated by multiplying the **DFT** of a time waveform by its **complex conjugate**.

A **Cross** spectrum is calculated by multiplying the **DFT** on one time waveform by the **complex conjugate** of the **DFT** of a different time waveform.

When the **FFT** is applied to a vibration signal, the **linear** part (typically due to **resonant** or **order-related** vibration) is **transformed to the same frequencies** in the spectrum. The **non-linear** part of the vibration is transformed into **random noise**, which is **spread throughout** the **DFT**. Random noise can be **averaged out** of an **Auto** or Cross spectrum by **summing together** multiple estimates of the spectrum.

DFT

A **DFT** is calculated by transforming a block (or sampling window) of uniformly-sampled time domain samples using the **FFT** algorithm. In MEscope, this is done by executing the **Transform | FFT** command in a Data Block window containing one or more time domain waveforms (**M#s**).

$$\mathbf{X(\omega) = FFT\big(x(t)\big)}$$

AVERAGING DFT ESTIMATES

With experimentally acquired data, it is assumed that a time waveform and its DFT are made up of the **sum of two parts**, the **intended signal** and **additive random noise**. When **multiple blocks** of the same signal are acquired, each block is assumed to contain **the same intended signal** plus **different additive random noise**.

The additive random noise is assumed to have a **Gaussian** or **Normal** distribution of magnitudes versus samples.

Therefore, when a number of **DFT** estimates of the same signal **are summed together**, the random noise will be reduced or nearly eliminated, and the **intended signal will be preserved** in the final result.

The **complex valued DFT contains two parts**, the intended signal term, (**A+jB**), and a noise term, (**C_i+jD_i**), which is assumed to change from each estimate (i). Therefore, the ith estimate of the **DFT** ($X_i(\omega)$) can then be written as,

$$\mathbf{X_i(\omega) = A(\omega) + jB(\omega) + C_i(\omega) + jD_i(\omega)}$$

AVERAGE DFT

An average **DFT** is calculated by averaging together several **DFT** estimates. The average **DFT** is calculated by averaging the real & imaginary parts of the **DFT** separately. The average **DFT** can be written as follows,

$$\begin{aligned} \mathrm{AveX}(\omega) &= \tfrac{1}{N}\sum_{i=1}^{N} X_i(\omega) \\ &= \tfrac{1}{N}\sum_{i=1}^{N}\left(A(\omega)+jB(\omega)+C_i(\omega)+jD_i(\omega)\right) \\ &= A(\omega)+jB(\omega)+\tfrac{1}{N}\sum_{i=1}^{N}\left(C_i(\omega)+jD_i(\omega)\right) \end{aligned}$$

This average is called **stable (or linear) averaging** in MEscope.

DIFFICULTY WITH THE AVERAGE DFT

The difficulty with averaging multiple estimates of the DFT is that the intended signal itself is not the same unless all of the time waveforms transform to the same (**A+jB**).

The same intended **DFT** (**A+jB**) can only be assured if the **same time waveform is repeated** in each sampling window.

If the time waveform is not repeated, then the **phase of the intended DFT** will average to **an incorrect value**.

AUTO SPECTRUM

The **Auto** spectrum is calculated by multiplying the **DFT** by its own **complex conjugate**.

The **Auto** spectrum has a magnitude only, and its phase is zero.

Therefore, the average of multiple **Auto** spectrum estimates is merely the average of multiple magnitudes, one from each **Auto** spectrum estimate.

AVERAGE AUTO SPECTRUM

In the equation below, the average **Auto** spectrum is expressed as a function of multiple **DFT** estimates, using **stable (or linear) averaging**.

The last term of the equation is the average magnitude of the noise. Since this **noise power term** is **always positive**, it **cannot be removed** by averaging multiple estimates.

$$\begin{aligned} X_{APS}(\omega) &= \tfrac{1}{N}\sum_{i=1}^{N} X_i(\omega)X_i^*(\omega) \\ &= \tfrac{1}{N}\sum_{i=1}^{N}\left(A(\omega)+jB(\omega)+C_i(\omega)+jD_i(\omega)\right)\left(A(\omega)-jB(\omega)+C_i(\omega)-jD_i(\omega)\right) \\ &= A^2(\omega)+B^2(\omega)+\tfrac{1}{N}\sum_{i=1}^{N}\left(2A(\omega)C_i(\omega)+2B(\omega)D_i(\omega)+C_i^2(\omega)+D_i^2(\omega)\right) \\ &= A^2(\omega)+B^2(\omega)+\frac{2A(\omega)}{N}\sum_{i=1}^{N}C_i(\omega)+\frac{2B(\omega)}{N}\sum_{i=1}^{N}D_i(\omega)+\frac{1}{N}\sum_{i=1}^{N}\left(C_i^2(\omega)+D_i^2(\omega)\right) \end{aligned}$$

REMOVING RANDOM NOISE FROM AN AUTO SPECTRUM

Since the noise is assumed to be Gaussian random noise, some real part terms (**C_i**) will be positive and some negative. With sufficient averaging, the **average of the C_i values will tend towards zero**. The same is true for the imaginary components (**D_i**). This is a characteristic of Gaussian random noise.

To summarize, the **average magnitude** of the noise term does not sum to zero, but the **average of the real & imaginary parts** of the noise **will sum to zero**.

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N} C_i(\omega)\right) = 0,\ \lim_{N\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N} C_i^2(\omega)\right) \neq 0$$

$$\lim_{N\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N} D_i(\omega)\right) = 0,\ \lim_{N\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N} D_i^2(\omega)\right) \neq 0$$

Removing the zero noise terms from the Equation for the **average Auto spectrum**,

$$X_{APS}(\omega) = A^2(\omega) + B^2(\omega) + \frac{1}{N}\sum_{i=1}^{N}\left(C_i^2(\omega) + D_i^2(\omega)\right)$$

This equation shows that **random noise cannot be removed** from an **average Auto spectrum**, no matter how many estimates are averaged together.

REMOVING RANDOM NOISE FROM A CROSS SPECTRUM

A **Cross** spectrum is the **DFT** of one time waveform multiplied by the complex conjugate of the **DFT** of a different time waveform.

The mathematics for an average Cross spectrum between a **DFT** of a signal with additive noise $(\mathbf{A}(\omega) + \mathbf{jB}(\omega)) + (\mathbf{C_i}(\omega) + \mathbf{jD_i}(\omega))$ and the conjugate of the **DFT** of a second signal with additive noise $(\mathbf{E}(\omega) - \mathbf{jF}(\omega)) + (\mathbf{G}(\omega) - \mathbf{jH}(\omega))$ is written as,

$$X_{XPS}(\omega) = \frac{1}{N}\sum_{i=1}^{N}\Big((A(\omega) + jB(\omega)) + (C_i(\omega) + jD_i(\omega))\Big)\Big((E(\omega) - jF(\omega)) + (G_i(\omega) - jH_i(\omega))\Big)$$

Multiplying together all of these terms gives,

$$X_{XPS}(\omega) = \Big((A(\omega)E(\omega) + B(\omega)D(\omega)) + j(B(\omega)E(\omega) - A(\omega)F(\omega))\Big) + \frac{1}{N}\sum_{i=1}^{N}(\text{linear noise terms})$$

Averaging will **reduce the linear noise terms to zero** leaving a complex-valued **average Cross spectrum** estimate as the final result.

Random noise is removed from an **average Cross spectrum** when many Cross spectrum estimates are averaged together.

CONCLUSIONS

- Multiple estimates of the **DFT** can be averaged together to remove random noise, but the will have an **incorrect phase** phase unless the time waveforms used for spectrum averaging contain **the same intended signal**. This is usually accomplished by using a trigger for data acquisition.
- The random noise **power is added** to an **average Auto spectrum** when multiple estimates are averaged together
- Random **noise is removed** from an **average Cross spectrum** when multiple estimates are averaged together