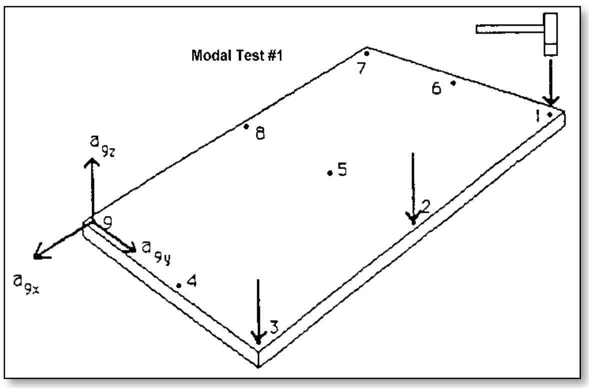
MEscope Application Note 05
Impact Testing Question

INTRODUCTION

There are two ways to perform a modal test on a structure using an impact hammer, a tri-axial accelerometer, and a 4-channel data acquisition system or analyzer. One test is called a ***roving impact test*** and the other is called a ***roving response (or roving accelerometer) test***.

Modal Test #1: Roving Impact Test

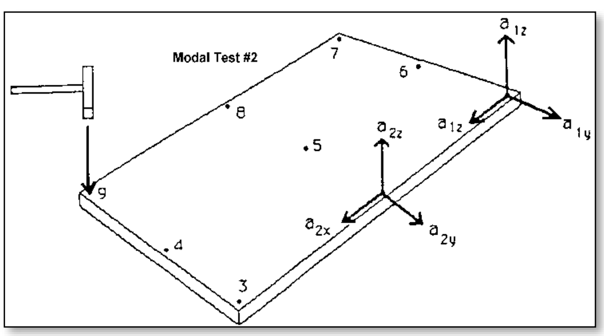
In a roving impact test, Frequency Response Function (FRF) measurements are made by attaching the tri-axial accelerometer at a fixed point on the structure (Point **9 below**), and impacting the structure at points **1** through **9** in the **Z** direction. Since three acceleration outputs are simultaneously measured for each force input, a total of **27 FRFs** are calculated from the acquired data.



Modal Test #1: Roving Impact Test.

Test #2: Roving Accelerometer Test

In a ***roving accelerometer test***, FRF measurements are made by repeatedly impacting the structure ***at the same point and in the same direction*** (**9Z** below). The tri-axial accelerometer is attached to point **1** for the first measurement and is then moved to point **2 for the second measurement**, point **3 for the third measurement**, and so on. Again, a total of **27 FRFs** are calculated from the acquired data.



Modal Test #2: Roving Accelerometer Test.

THE QUESTION

If both methods are applied to a structure with ***nine test points*** of interest, do the two test methods yield the same modal information?

Even though the test article looks like a flat plate, this question ***applies to testing any structure***. The Z direction can be any direction you choose, and it can be *a **different direction at each test point***.

THE ANSWER

The two modal tests ***do not yield the same modal information***. Modal Test #1 will provide mode shapes with only nine degrees-of-freedom (DOFs). Each mode shape will contain motion ***only in the Z direction*** at the nine test points.

Modal Test #2 will provide mode shapes that exhibit ***motion in three directions (X, Y & Z)*** at each of the nine test points.

On the surface, Modal Test #1 appears to offer little merit. However, this test is a ***multiple reference test***, and provides ***three estimates of each mode shape***. All three mode shape estimates for each mode should be the same, thus confirming that a true resonance has been excited. On the other hand, Modal Test #2 ***is a single reference test,*** and only provides one reference of data corresponding to the fixed DOF of the impact hammer.

The FRFs calculated from the data acquired in Modal Test #1 can be curve fit using a ***multiple reference curve fitting method***. A multiple reference method can estimate modal parameters for ***closely coupled modes*** or ***repeated roots***. ***Local modes*** that are segregated to local regions of the structure can also be identified from a multiple reference set of FRFs. FRFs from a single reference test like Modal Test #2 cannot be used in this manner.

In summary, Modal Test #1 is a ***multiple reference modal test*** yielding ***three estimates of each mode shape***, but with only 1 DOF of motion (in the Z direction) at each of the nine test points. Modal Test #2 is a ***single reference modal test*** yielding mode shapes with 3 DOFs of motion (in the X, Y & Z directions) at each test point.

DETAILS BEHIND THE ANSWER

An experimental modal analysis characterizes the dynamic properties between **N** degrees-of-freedom (DOFs) of a structure. Each DOF of a mode shape defines the ***motion** at a **specific point on a structure in a specific direction.***

There are *N^2 possible FRFs* that could be measured between pairs of the **N** DOFs on a structure. These FRFs can be arranged in an **N** by **N** square matrix, Each FRF is a function of frequency, and defines the motion (displacement, velocity, or acceleration) at a DOF per unit of force applied at another DOF.

For example, to completely characterize the dynamics between 100 DOFs of a structure with an FRF matrix, ***10,000 FRFs would be required***. That would render modal testing, or Experimental Modal Analysis (EMA), ***very impractical***. ***It would be too time consuming!***

In general, it is ***not necessary*** to measure all $\mathbf{N^2}$ possible FRFs between N DOFs of a structure. The dynamics of most structures can be completely characterized by measuring just *a **single row*** or *a **single column*** of the FRF matrix. Therefore, only **N** FRFs normally need to be measured.

The reason for this is that FRFs can be represented in terms of modal parameters, which in themselves completely characterize the dynamics of a structure.

Modal analysis is a mathematical way of defining the resonant vibration of a structure in terms of its natural resonances, or modes of vibration.

Each mode of vibration is defined by *three* parameters

Modal frequency

Modal damping

Mode shape

NOTE: When a set of modes is used in **Structural Dynamics Modification (SDM)** to predict the effects of physical modifications to a structure, a fourth modal parameter (called ***modal mass***) is also required.

A ***broad-band modal test*** such as an impact test is performed to excite enough modes and acquire enough data and identify all of the modes in a wide band of frequencies from a set of FRF measurements. Every FRF will exhibit characteristics that reflect the frequency & damping of those resonances that were excited by the broad-band impact force. A set of **N** FRFs can be curve fit to identify the frequency, damping, and mode shape associated with each resonance. Each mode shape will have **N** unique elements (or components) in it, one for each DOF measured.

LOCAL MODES

Often, a single FRF will not contain evidence (a resonance peak) of all of the modes that may exist in the bandwidth of the FRF measurements. Modes that only have resonance peaks in a few FRFs are referred to as ***local modes***. When a structure contains local modes, several (fixed) reference DOFs must be chosen in order to excite and identify all of the local modes from the resulting FRFs.

CLOSELY COUPLED MODES

More than **N** FRFs may also be required from structures with ***closely coupled modes***. Two or more closely coupled modes have ***nearly the same modal frequency*** and ***only exhibit one resonance peak in the FRFs***. For structures with closely coupled modes, ***two or more rows or columns*** of the **FRF** matrix must be measured and curve fit using ***multiple reference curve fitting*** to identify the closely coupled modes.

REPEATED ROOTS

Structures with *spatial symmetry* often have ***repeated roots in them.*** A repeated root is ***two or more modes with the same modal frequency, but with different mode shapes***. For structures with repeated roots, ***two or more rows or columns*** of the **FRF** matrix must be measured and curve fit using ***multiple reference curve fitting*** to identify the repeated roots.

FRFS IN TERMS OF MODAL PARAMETERS

The FRF matrix can be written in **partial fraction expansion form** as a ***summation of pairs of terms***, each pair of terms containing the contribution of a single mode of vibration.

$$\left[H_{ij}(j\omega)\right]=\sum_{k=1}^{M}\frac{1}{2j}\left[\frac{\left[R_{ij}(k)\right]}{\left(j\omega-p_k\right)}-\frac{\left[R_{ij}^{*}(k)\right]}{\left(j\omega-p_k^{*}\right)}\right]$$

Where $H_{ij}(j\omega)$ = the element in the **i**th row and **j**th column of the FRF matrix

Each FRF represents the motion (displacement, velocity, or acceleration) at DOF(**i**) per unit of force applied at DOF(**j**). **FRFs have** typical engineering units of m/N or in/lb.

p_k = the pole for mode(k) = $-\sigma_k + j\omega_k$ (rad/sec)

ω_k = damped natural frequency of mode(k)

σ_k = damping decay of mode(k). (rad/sec)

$R_{ij}(k)$= Residue between DOF(**i**) and DOF(**j**) for mode(k) (m/N-sec or in/lb-sec)

ω = the forcing frequency (or independent variable) (rad/sec)

$j=\sqrt{-1}$ and ***** indicates complex conjugation

M = the number of modes

Each FRF is a summation of **2M** terms. Each term contains a ***residue divided by a pole***. The ***denominator of all FRFs is the same***. Each term in the summation contains the same pair of poles for each mode(**k**).

Only the numerators of the FRFs *are* ***different***. Each numerator contains a specific residue, $R_{ij}(k)$, which is dependent upon the response DOF(**i**) and the excitation DOF(**j**), and is different for each mode(**k**).

Residues: Residues are ***physical properties*** of a structure. They are referred to as the ***strength*** or ***participation*** of a mode between two structural DOFs. They also define the ***height of each resonance peak*** in an FRF. Residues have the same ***units as the FRF multiplied by (radians/second).***

The *Residue matrix* for each mode(k) is an (**N** by **N)** array of its residues. Normally, the FRF matrix is assumed to be ***symmetric***. This follows from the following assumption, namely that most structures exhibit ***dynamic*** *reciprocity*.

Dynamic Reciprocity: The FRF between a force input at DOF(A) and its resulting response at DOF(B) ***equals*** the FRF between a force input at DOF(B) and its resulting response at DOF(A).

If the FRF matrix is assumed to be symmetric, then the Residue matrix is also symmetric.

In a ***single-reference modal test, a single row or column*** of FRFs is calculated from the acquired data. Hence the same ***row or column*** of the Residue matrix for each mode is obtained by curve fitting the FRFs.

In a ***multiple-reference modal test**, **multiple rows or columns*** of FRFs are calculated from the acquired data. Hence the same ***rows or columns*** of the Residue matrix for each mode are obtained by curve fitting the FRFs.

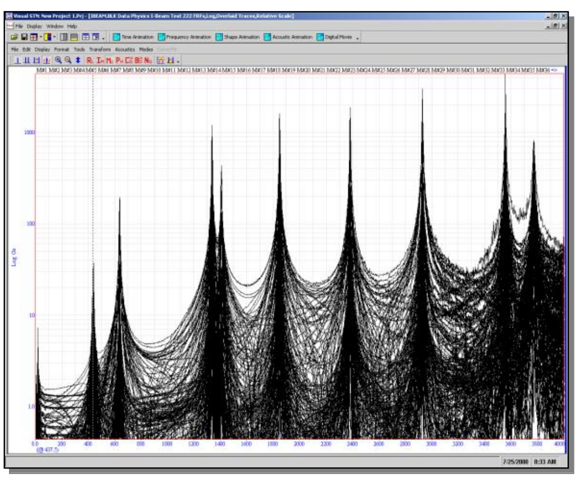
It is shown later ***that each row or column of the Residue matrix*** contains the same *mode shape* for each resonance that is represented in the set of FRFs. ***One row or column of FRFs*** (and therefore residues) is usually sufficient to indentify the mode shapes of all of the modes with frequencies in the bandwidth of the FRFs.

GLOBAL MODES

Since the ***same pole*** (modal frequency & damping) is contained ***in all*** FRFs acquired from the same structure, all of the **M** modal frequencies & damping could be identified from a *single* FRF. A larger number of N FRFs is acquired solely to identify the residues for each mode, each row or column of which contains the same mode shape.

Global Pole Property: The ***same pole (modal frequency & damping)*** for each mode ***is contained in every FRF*** that is acquired from a structure.

Each modal residue defines the strength or ***height of the resonance peak*** for that mode in each FRF. Hence poles are easier to identify from some FRFs than in others, depending on the height of a resonance peak relative to the other peaks in the FRF. In general, the best estimate of each pole will be obtained from FRFs that have large resonance peaks (large residues) associated with that pole.

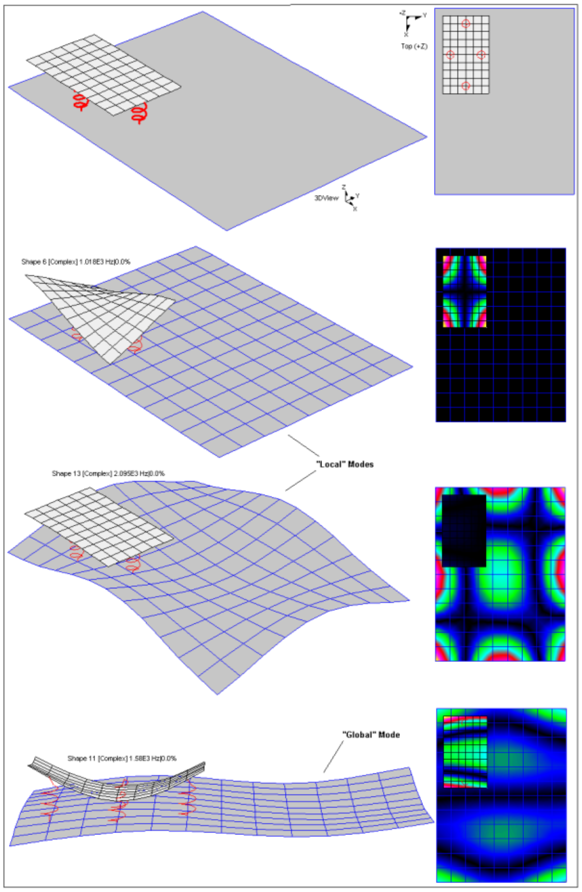


Overlaid FRF Magnitudes showed global poles.

The figure above shows the magnitudes of a number of FRFs overlaid on one another. Notice that the resonance peak in each FRF appears at the same frequency. This shows that the modal frequency for each resonance (approximately the frequency of the resonance peak), is the same no matter where the FRFs were measured on the structure. This is the ***global property*** of modal frequencies.

LOCAL MODES

It is not uncommon to encounter ***local mode***s in structures that are made up of different components,. The combination of two plates joined together by four springs (shown below) is a simple example of such a structure. The small plate exhibits local modes that do not participate in the motion of the base plate below it.

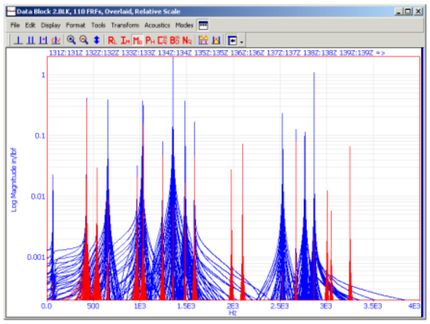


Local modes in a two-plate structure.

Trapped Energy: Local modes result when excitation energy ***becomes trapped*** in a local region of a structure and is not readily dissipated throughout the structure.

Local modes can also be deliberately design into a structure. In this simple plate example, the four mounting springs were placed on the centerlines of the small plate. These locations are ***nodal points*** in the Z-axis (vertical) direction for many of the modes of the small plate.

The FRFs from the small plate (show in red below) contain different frequencies from those in the FRFs of the base plate (shown in blue).



FRFs show different resonance peaks of local modes.

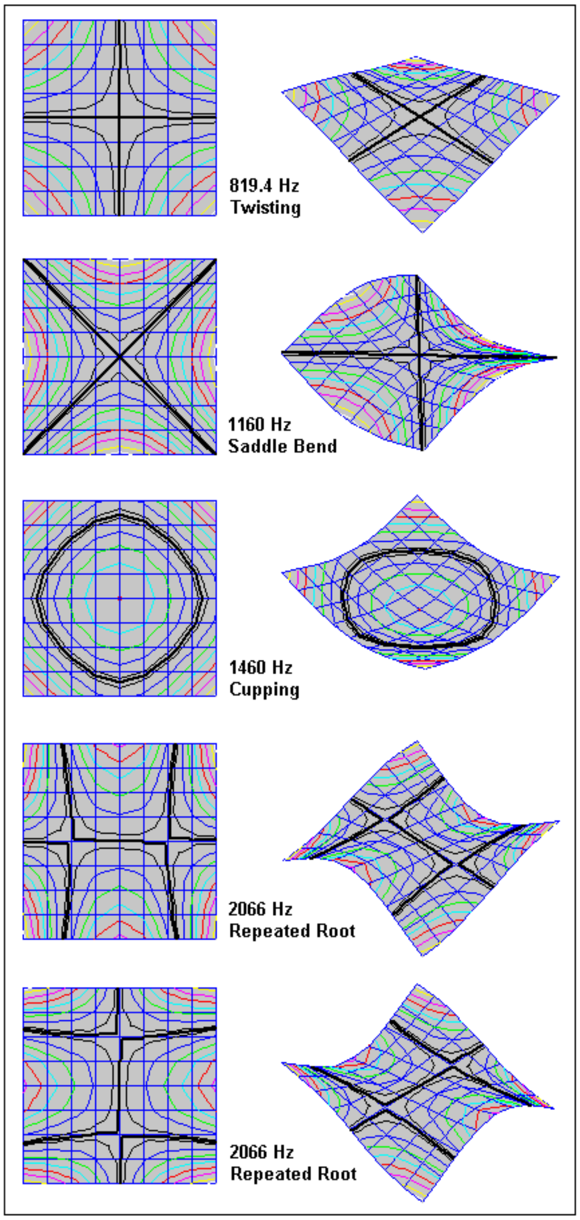
MULIPLE REFERENCE MODAL TEST

When **local modes are suspected or encountered** in a structure, a ***multiple reference modal test*** is required to ensure that no modes will be missed. For example, a single reference modal test on the ***base plate*** of the previous two-plate structure example would fail to capture the ***local modes*** of the small plate mounted above the base plate.

Likewise, performing a single reference modal test on the small plate would identify its modes, but some ***local modes*** of the base plate would be missed. A ***single reference* test** would capture the *global modes* common to both plates, but depending on the reference used, would miss some of the local modes of one of the plates.

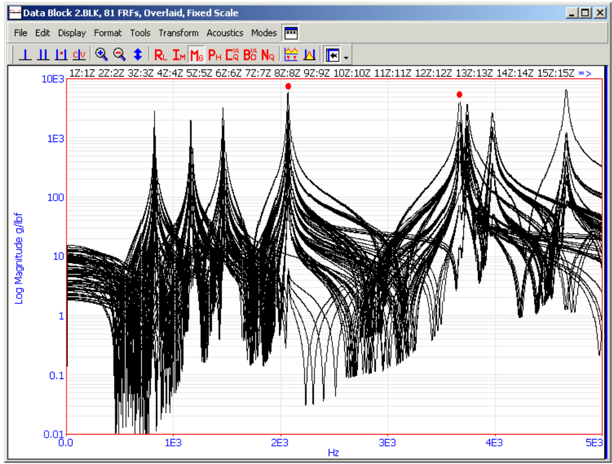
Repeated Roots of a Square Plate

Multiple reference modal testing is usually required when a structure is very simple but ***geometrically symmetrical***. Consider the first five modes of the square plate with free-free boundary conditions shown below. The first three modes have ***unique poles*** which can be found by ***single reference modal testing***, ***but*** the ***fourth & fifth modes*** are a ***pair of repeated-roots***.



A square plate with repeated roots.

A square plate like the one above will contain ***many pairs of repeated roots***. The mode shape for one of the repeated roots is identical to the mode shape of the other, but is rotated about the axis of symmetry, in this case the **Z** -axis. Repeated roots of ***multiplicity higher than two*** are also possible but are rarely encountered in real structures.



Driving-point FRFs from square plate.

The experimental FRFs from a structure with repeated roots give no direct indication of repeated roots. The ***red dots*** on the FRFs above indicate ***repeated roots*** of the square plate, and the other resonance peaks indicate ***six distinct poles***. Multi-Reference curve fitting of multiple rows or columns of FRFs from the FRF matrix is required to correctly extract the mode shapes of repeated roots.

MODE SHAPES VS. OPERATING DEFLECTION SHAPES

The measured motion of **N** DOFs of any structure can be arranged in vector form as,

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ : \\ x_i \\ x_N \end{Bmatrix}$$

Where $\mathbf{x_i}$ is the motion of DOF(i) of the structure. This vector is called an ***Operating Deflection Shape*** **(ODS)** or simply a deflection shape.

Principles of Resonant Vibration

Two principles govern the way resonant vibration can be represented in terms of mode shapes.

1. All resonances (or modes of vibration) of a structure are ***excited at all frequencies***
2. All resonant vibration is made up of a ***linear combination of mode shapes***

Each mode shape (with **N** DOFs) can be written as a vector (with N components in it). The mode shape $\{u_k\}$ of mode(k) can be written as:

$$\{u_k\} = \begin{Bmatrix} u_{1k} \\ u_{2k} \\ : \\ u_{ik} \\ u_{Nk} \end{Bmatrix}$$

Where u_{ik} is the motion of the DOF(i) of mode(k). The mode shapes of **M** modes can be arranged as ***columns of*** an (M by N) ***mode shape matrix*** [U],

$$[U]=\begin{bmatrix} u_{11} & u_{12} & .. & u_{1k} & u_{1M} \\ u_{21} & u_{12} & .. & u_{2k} & u_{2M} \\ u_{31} & u_{12} & .. & u_{3k} & u_{3M} \\ : & : & .. & : & : \\ u_{N1} & u_{12} & .. & u_{Nk} & u_{NM} \end{bmatrix}$$

Second Principle of Resonant Vibration: If the mode shapes are ***linearly independent*** of one another, ***any motion*** of a structure that is represented by an ODS can also be represented as a linear combination of the mode shapes of the structure.

This principle is stated by the following equation,

$$\{x\}=[U]\{q\}$$

Where $\{q\}$ is an M-vector of ***modal participation factors****,* also called ***generalized coordinates***. The modal participation factors are unique and the above equation can be solved for them ***if and only if the mode shapes are linearly independent*** of one another. If the mode shapes are linearly independent of one another, any ODS can be uniquely represented in terms of mode shapes.

DIFFERENTIAL EQUATIONS OF MOTION

In Experimental Modal Analysis (EMA), we start with a set of experimental FRF and extract a different model of the structure in terms of its modal parameters, called a ***modal model***. In Finite Element Analysis (FEA), we start with a set of differential equations of motion for the structure form which a modal model can also be extracted.

Both FRFs and differential equations of motion represent the same dynamic properties of a structure. The modal model extracted from either the FRFs or the differential equations also represents the same dynamic properties of the structure.

FEA is used to create three coefficient matrices used in a finite number of coupled differential equations to define the motion between N DOFs of a structure. These three (**N** by **N)** matrices contain the mass [M], damping [C] and stiffness [K] properties of the structure. These matrices, together with the N-vector {F} of the externally applied forces to the structure, are arranged as the set of ***linear second order differential equations***, shown below

$$[M]\{\ddot{x}\}+[C]\{\dot{x}\}+[K]\{x\}=\{F\}$$

$\{\ddot{x}\}$ = an N-vector of the acceleration at each DOF

$\{\dot{x}\}$= an N-vector of the velocity at each DOF

$\{x\}$ = an N-vector of the displacement at each DOF

[M], [C] and [K] are *symmetric* because the structure is assumed to exhibit ***Dynamic Reciprocity***. The stiffness matrix [K] is also assumed to be ***positive definite***, meaning than the structure does not require external forces to hold it in position.

If the ***external forces*** on the right-hand side of the equations of motion ***are zero***, the differential equations can still be solved for ***non-trivial solutions***. As many as N ***non-trivial solutions*** exist, and they are called eigensolutions. Each eigensolution consists of an eigenvalue and an eigenvector. Each eigenvalue is a modal frequency and each eigenvector is a mode shape.

The mode shapes exhibit ***orthogonality*** with respect to both the mass [M] and stiffness [K] matrices. Mass matrix ***orthogonality*** is written

$$\{u_k\}^T [M] \{u_k\} = m_k$$

Where $\mathbf{m_k}$ is called the ***modal mass*** of mode(k), and

$$\{u_k\}^T [M] \{u_m\} = \{u_m\}^T [M] \{u_k\} = 0$$

Where mode(m) is a different mode than mode(k)

Using this ***orthogonality property*** (and assuming that the damping matrix [**C**] is [**0**]), the differential equations can be re-written in a ***diagonalized*** form

$$\begin{aligned} & [U]^T [M][U]\{\ddot{q}\} + [U]^T [K][U]\{q\} = [U]^T \{F\} \\ & = \left[\ddots m_k\right]\{\ddot{q}\} + \left[\ddots m_k \omega_k^2\right]\{q\} = [U]^T \{F\} = \{Q\} \end{aligned}$$

Where $[U]^T$ is the matrix transpose of [U] and the vector {Q} is called a ***generalized force***. The diagonal elements (m_k) are the ***modal masses***, and are also called ***generalized masses.***

For lightly damped structures, it can also be assumed that the ***damping matrix is diagonalized*** by the mode shapes. Hence three new terms, ***modal mass, modal damping, & modal stiffness*** are defined by the equations

$$[U]^T [M][U] \cong \left[\ddots m_k\right]$$

$$[U]^T [C][U] \cong \left[\ddots 2 m_k \sigma_k\right]$$

$$[U]^T [K][U] \cong \left[\ddots m_k \left(\sigma_k^2 + \omega_k^2\right)\right]$$

Unit Modal Mass Scaling

An important property of mode shapes is that they are eigenvectors, meaning that ***their "shape" is unique*** but their values are not. However, the equation above shows that each modal mass and its associated mode shape are related to one another. Both are arbitrary, but if a value of one is chosen, the value of the other is then fixed.

It is convenient to scale each mode shape so that its associated modal mass is equal to **"1" mass unit** (1 kg, 1 lb_m, etc.). Mode shapes that are scaled in this manner are called **UMM mode shapes in MEscope.**

RESIDUES AND MODE SHAPES

It has already been shown that ***residues*** are the numerator terms in the analytical expression for an FRF. Therefore, residues retain the physical units of the FRFs obtained from an experimental modal analysis (EMA). Recall that the units of residues are the ***FRF units multiplied by (radians/second)***. The residues associated with each mode are ***physical constants*****,** unlike mode shapes that can be arbitrarily scaled.

Each element of the residue matrix for mode(k) is equal to the ***product of two DOFs of its mode shape*** divided by its *modal mass*. Because of ***dynamic reciprocity***, the FRF matrix is symmetrical, and therefore the residue matrix is also symmetrical.

$$R_{ij}(k) = \frac{u_{ik}u_{jk}}{m_k} = \frac{u_{jk}u_{ik}}{m_k} = R_{ji}(k)$$

u_{ik} = mode shape component for DOF(**i) of** mode(k)

u_{jk} = mode shape component for DOF(**i) of** mode(k)

m_k = modal mass of mode(k).

This important relationship between residues and mode shapes is what makes modal testing (or EMA) practical. It can be stated as follows,

> **Modal Testing Assumption:** Every row & column of the residue matrix for a mode contains its mode shape. Therefore, any row or column of the FRF matrix can be measured and curve fit to yield the same row or column of residues, which contains the mode shape

Every row and every column of the residue matrix contains the mode shape multiplied by a different DOF of the mode shape. This strong result makes modal testing and EMA practical.

ANSWER TO THE QUESTION

For **Modal Test #1** (***the roving impact test***), the response sensor remained at a fixed location, oriented in fixed directions. Using a single axis sensor, with each new impact point, another element in ***one row of the FRF matrix*** is calculated, and hence the same row of the residue matrix is calculated.

Since a tri-axial accelerometer was used in Modal Test #1, there are *three fixed response DOFs*, hence FRFs in ***three rows*** of the FRF matrix are calculated for each impact point and residues in ***three rows*** of the Residue matrix are calculated for each mode.

For **Modal Test #1**, the calculated FRFs and residues, and the mode shape components derived from the residues, are tabulated below.

FRFs		Residues		Mode Shape
$H_{9x,1z}$	$\rightarrow$	$R_{9x,1z} = u_{9x}\, u_{1z}$	$\rightarrow$	u_{1z}
$H_{9y,1z}$	$\rightarrow$	$R_{9y,1z} = u_{9y}\, u_{1z}$	$\rightarrow$	u_{1z}
$H_{9z,1z}$	$\rightarrow$	$R_{9z,1z} = u_{9z}\, u_{1z}$	$\rightarrow$	u_{1z}
$H_{9x,2z}$	$\rightarrow$	$R_{9x,2z} = u_{9x}\, u_{2z}$	$\rightarrow$	u_{2z}
$H_{9y,2z}$	$\rightarrow$	$R_{9y,2z} = u_{9y}\, u_{2z}$	$\rightarrow$	u_{2z}
$H_{9z,2z}$	$\rightarrow$	$R_{9z,2z} = u_{9z}\, u_{2z}$	$\rightarrow$	u_{2z}
•	•			•
•	•			•
•	•			•
$H_{9x,9z}$	$\rightarrow$	$R_{9x,9z} = u_{9x}\, u_{9z}$	$\rightarrow$	u_{9z}
$H_{9y,9z}$	$\rightarrow$	$R_{9y,9z} = u_{9y}\, u_{9z}$	$\rightarrow$	u_{9z}
$H_{9z,9z}$	$\rightarrow$	$R_{9z,9z} = u_{9z}\, u_{9z}$	$\rightarrow$	u_{9z}

The residue DOFs are the same as the FRF DOFs, so ***three rows of the residue matrix*** corresponding to the three DOFs of the tri-axial accelerometer would be arranged as shown below.

Residue Matrix

$$
\begin{array}{c}
\begin{array}{cccccc} \mathbf{1Z} & \mathbf{2Z} & \mathbf{3Z} & & \bullet\;\bullet\;\bullet & \mathbf{9Z} \end{array} \\
\left[\begin{array}{cccccc}
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
 & & & & & \\
R_{9X,1Z} & R_{9X,2Z} & R_{9X,3Z} & & \bullet\;\bullet\;\bullet & R_{9X,9Z} \\
R_{9Y,1Z} & R_{9Y,2Z} & R_{9Y,3Z} & & \bullet\;\bullet\;\bullet & R_{9Y,9Z} \\
R_{9Z,1Z} & R_{9Z,2Z} & R_{9Z,3Z} & & \bullet\;\bullet\;\bullet & R_{9Z,9Z}
\end{array}\right]
\begin{array}{c} 1X \\ 1Y \\ 1Z \\ \bullet \\ \bullet \\ \bullet \\ 9X \\ 9Y \\ 9Z \end{array}
\end{array}
$$

The residues yield *three estimates* of each mode shape, and each shape has 9 DOFs **1Z, 2Z, … 9Z**. Each residue matrix row contains sufficient information to calculate the ***modal frequency, damping & mode shape of a mode***, but only the ***third row contains a driving-point*** FRF (**9Z:9Z**)

UMM mode shapes can be calculated by scaled the row of residues containing the ***driving point*** Residue (**9Z:9Z**). A modal model containing **UMM mode shapes** can be used with ***Structural Dynamics Modification* (SDM**) to explore physical modifications to the structure.

For **Modal Test #2**, (***the roving accelerometer test***), the excitation DOF was fixed. Each time the tri-axial accelerometer was moved to a new point, three new FRFs in the column corresponding to the impact force were calculated. Three new residues would be calculated from these FRFs, and three new mode shape components would be the result for each mode. Hence, mode shapes with 27 DOFs are obtained for each mode in the bandwidth on the FRFs.

For **Modal Test #2**, the calculated FRFs and residues, and the mode shape components derived from the residues, are tabulated below.

FRFs		**Residues**		**Mode Shape**
$H_{1x,9z}$	$\rightarrow$	$R_{1x,9z} = u_{1x}\, u_{9z}$	$\rightarrow$	u_{1x}
$H_{1y,9z}$	$\rightarrow$	$R_{1y,9z} = u_{1y}\, u_{9z}$	$\rightarrow$	u_{1y}
$H_{1z,9z}$	$\rightarrow$	$R_{1z,9z} = u_{1z}\, u_{9z}$	$\rightarrow$	u_{1z}
$H_{2x,9z}$	$\rightarrow$	$R_{2x,9z} = u_{2x}\, u_{9z}$	$\rightarrow$	u_{2x}
$H_{2y,9z}$	$\rightarrow$	$R_{2y,9z} = u_{2y}\, u_{9z}$	$\rightarrow$	u_{2y}
$H_{2z,9z}$	$\rightarrow$	$R_{2z,9z} = u_{2z}\, u_{9z}$	$\rightarrow$	U_{2z}
		• •	•	
		• •	•	
		• •	•	
$H_{9x,9z}$	$\rightarrow$	$R_{9x,9z} = u_{9x}\, u_{9z}$	$\rightarrow$	u_{9x}
$H_{9y,9z}$	$\rightarrow$	$R_{9y,9z} = u_{9y}\, u_{9z}$	$\rightarrow$	u_{9y}
$H_{9z,9z}$	$\rightarrow$	$R_{9z,9z} = u_{9z}\, u_{9z}$	$\rightarrow$	u_{9z}

The residue DOFs are the same as the FRF DOFs, so ***one column of the residue matrix*** corresponding to the impact point DOF would be arranged as shown below.

<u>Residue Matrix</u>

$$
\begin{array}{cc}
\begin{array}{ccccccc} 1Z & 2Z & 3Z & \bullet & \bullet & \bullet & 9Z \end{array} & \\
\left[\begin{array}{ccccccc}
 & & & & & & R_{1X,9Z} \\
 & & & & & & R_{1Y,9Z} \\
 & & & & & & R_{1Z,9Z} \\
 & & & & & & R_{2X,9Z} \\
 & & & & & & R_{2Y,9Z} \\
 & & & & & & R_{2Z,9Z} \\
 & & & & & & \bullet \\
 & & & & & & \bullet \\
 & & & & & & \bullet \\
 & & & & & & R_{9X,9Z} \\
 & & & & & & R_{9Y,9Z} \\
 & & & & & & R_{9Z,9Z}
\end{array}\right] &
\begin{array}{c} 1X \\ 1Y \\ 1Z \\ 2X \\ 2Y \\ 2Z \\ \bullet \\ \bullet \\ \bullet \\ 9X \\ 9Y \\ 9Z \end{array}
\end{array}
$$

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